

Consider the homogeneous linear differential equation $4y'' + 12y' + By = 0$.

SCORE: ____ / 5 PTS

- [a] Find the general solution if $B = 9$.

$$\begin{aligned} 4r^2 + 12r + 9 &= 0 \\ (2r + 3)^2 &= 0 \end{aligned}$$

$$r = -\frac{3}{2}, -\frac{3}{2}$$

$$y = Ae^{-\frac{3}{2}x} + Bxe^{-\frac{3}{2}x}$$

- [b] Find the general solution if $B = 25$.

$$\begin{aligned} 4r^2 + 12r + 25 &= 0 \\ (2r + 3)^2 + 16 &= 0 \end{aligned}$$

$$r = \frac{-3 \pm 4i}{2} = -\frac{3}{2} \pm 2i$$

$$y = Ae^{-\frac{3}{2}x} \cos 2x + Be^{-\frac{3}{2}x} \sin 2x$$

Use the Wronskian to prove that the functions $f_1(x) = x^a \cos(b \ln x)$ and $f_2(x) = x^a \sin(b \ln x)$ are linearly independent on $(0, \infty)$ if $a, b \in \mathbb{R}$ and $b \neq 0$.

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$$\begin{aligned} & \begin{vmatrix} x^a \cos(b \ln x) & x^a \sin(b \ln x) \\ \underline{ax^{a-1} \cos(b \ln x) - bx^{a-1} \sin(b \ln x)} & \underline{ax^{a-1} \sin(b \ln x) + bx^{a-1} \cos(b \ln x)} \end{vmatrix} \\ &= \cancel{ax^{2a-1} \cos(b \ln x) \sin(b \ln x)} + bx^{2a-1} \cos^2(b \ln x) \\ &\quad - \cancel{ax^{2a-1} \cos(b \ln x) \sin(b \ln x)} + bx^{2a-1} \sin^2(b \ln x) \\ &= \underline{bx^{2a-1}} \neq 0 \end{aligned}$$

Find the general solution of the homogeneous linear differential equation $3x^2 y'' + xy' + y = 0$.

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$$3r^2 + (1-3)r + 1 = 0$$

$$\underline{3r^2 - 2r + 1 = 0} \quad \textcircled{1}$$

$$r = \frac{2 \pm \sqrt{-8}}{6} = \underline{\frac{1}{3} \pm \frac{\sqrt{2}}{3}i} \quad \textcircled{1}$$

$$y = \underline{A x^{\frac{1}{3}} \cos\left(\frac{\sqrt{2}}{3} \ln x\right) + B x^{\frac{1}{3}} \sin\left(\frac{\sqrt{2}}{3} \ln x\right)} \quad \textcircled{2}$$

$y_1 = \sin x$ is a solution of $(\tan x)y'' - 3y' + (3\cot x + \tan x)y = 0$.

Find a second linearly independent solution.

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$$y_2 = \underline{v \sin x} \quad (1)$$

$$y_2' = \underline{v' \sin x + v \cos x} \quad (1)$$

$$y_2'' = \underline{v'' \sin x + 2v' \cos x - v \sin x} \quad (1)$$

$$y_2 = \underline{\sin x \cos x} \quad (1)$$

$$(\tan x)(v'' \sin x + 2v' \cos x - v \sin x)$$

$$- 3(v' \sin x + v \cos x)$$

$$+ (3\cot x + \tan x)(v \sin x) \quad (2)$$

$$= \underline{v'' \sin x \tan x + v'(2 \sin x - 3 \sin x) + v(-\cancel{\sin x \tan x} - 3 \cos x + 3 \cos x + \cancel{\sin x \tan x})}$$

$$= 0 \rightarrow v'' \tan x - v' = 0 \quad \text{LET } u = v'$$

$$u' \tan x - u = 0$$

$$\frac{1}{2} \int \frac{1}{u} du = \int \cot x dx$$

$$\frac{1}{2} \int \frac{1}{u} du = \int \cot x dx \quad \ln|u| = \ln|\sin x|$$

$$\frac{1}{2} \int \frac{1}{u} du = \int \cot x dx \quad u = \sin x = v'$$

$$\frac{1}{2} \int \frac{1}{u} du = \int \cot x dx \quad v = -\cos x$$

Consider the non-homogeneous linear differential equation $3x^2y'' + xy' - 8y = Ax^2$.

SCORE: ____ / 8 PTS

- [a] If $y = x^2 \ln x$ is a particular solution of the equation, find the value of A .

$$y' = 2x \ln x + x \quad \left(\frac{1}{2}\right)$$

$$y'' = 2 \ln x + 2 + 1 = 2 \ln x + 3 \quad \left(\frac{1}{2}\right)$$

$$3x^2(2 \ln x + 3) + x(2x \ln x + x) - 8(x^2 \ln x) = 6x^2 \ln x + 9x^2 + 2x^2 \ln x + x^2 - 8x^2 \ln x = 10x^2 \quad \left(\frac{1}{2}\right)$$

$$A = 10 \quad \left(\frac{1}{2}\right)$$

- [b] Using superposition and linearity, find the general solution of $3x^2y'' + xy' - 8y = 5x^2$. $\leftarrow = \frac{1}{2}(10x^2)$

$$3r^2 + (1-3)r - 8 = 0$$

$$3r^2 - 2r - 8 = 0 \quad \left(\frac{1}{2}\right)$$

$$(3r+4)(r-2) = 0$$

$$r = -\frac{4}{3}, 2 \quad \left(\frac{1}{2}\right)$$

$$y = Ax^{-\frac{4}{3}} + Bx^2 + \frac{1}{2}x^2 \ln x \quad \left(\frac{1}{2}\right)$$

- [c] Solve the initial value problem, $3x^2y'' + xy' - 8y = 5x^2$, $y(1) = 5$, $y'(1) = 3$.

$$5 = A + B \quad \left(\frac{1}{2}\right)$$

$$3 = -\frac{4}{3}A + 2B + \frac{1}{2} \quad \left(\frac{1}{2}\right)$$

$$18 = -8A + 12B + 3$$

$$40 = 8A + 8B$$

MULTIPLY BY 6

$$58 = 20B + 3 \rightarrow B = \frac{55}{20} = \frac{11}{4}$$

$$A = 5 - B = \frac{9}{4}$$

$$y = \frac{9}{4}x^{-\frac{4}{3}} + \frac{11}{4}x^2 + \frac{1}{2}x^2 \ln x \quad \left(\frac{1}{2}\right)$$